

FLEXURAL DEFORMABILITY OF REINFORCED CONCRETE BEAMS

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ABSTRACT: In reinforced concrete beams, the sectional definition of the moment-curvature relationship is not straightforward owing to the presence of cracks. In general, this problem has been solved by considering a representative portion of the beam and by defining the curvature $1/r$ as the ratio between the rotation of the portion considered and its length. Large portions of the beam with many cracks lead to "average" moment-curvature relationship, while small portions delimited by two consecutive flexural cracks lead to "local" moment-curvature relationship. The difference between these two definitions is studied in this paper introducing a "general model." With this model the entire beam is modeled through a succession of blocks divided by flexural cracks taking into account the bond between steel and concrete. Due to the practical impossibility of univocal definition of the crack pattern evolution, it is proposed to use a "range model." This "range," delimited by the curves of maximum and minimum deformability, includes the moment-curvature relationship (locals and average) obtained from all possible crack patterns.

INTRODUCTION

The problem of reinforced concrete (RC) beams deformability has been addressed by most authors on the basis of the moment-curvature relationship. As is well known, this approach, which is the favorite tool of the "engineering beam theory," establishes a correlation within a generic cross section of the member between a single mechanical action (i.e., the bending moment M) and the corresponding deformation (i.e., the curvature $1/r$).

In RC beams, the sectional definition of this relationship is not straightforward owing to the presence of cracks. How can a single representative cross section be identified, if the opening of cracks gives rise to a complex configuration (Fig. 1) involving cracked sections (Sections ②), uncracked sections with no slip between steel and concrete (Sections ①), as well as sections that are still uncracked but exhibit slip (Sections ①-②)?

In general, this problem has been solved by considering a representative portion of the beam and by defining the curvature $1/r$ as the ratio between the rotation Φ of the portion considered and its length L .

In early studies, the reference portions of the beam were wide and contained many cracks. Their effects on the moment-curvature diagrams, $M-\Phi/L$ (of a type that might be defined as "average"), were therefore markedly smeared and gave little evidence for the formation of each crack (Fig. 2).

To account for these consequences, some authors [for instance, Gelfi and Giuriani (1982)] have introduced a "local" definition of the curvature, that is, referring to a suitable portion around each crack. The moment-curvature diagrams obtained from these two definitions may differ to a considerable extent, as clearly shown in Fig. 2.

In general, it should be noted that, in beams characterized by a virtually constant bending moment, average-type relationships may provide an accurate representation of physical behavior. On the other hand, local-type relationships work bet-

ter in portions of the beam characterized by steep gradients of the bending moment (Giuriani and Sforza 1981). Due to the characteristics of the testing equipment employed (displacement transducers and extensometers), in the past wide beam portions were generally considered, giving rise to average-type $M-\Phi/L$ diagrams. The data supplied by experimental investigations were used to calibrate most of the early sectional moment-curvature models described in the literature.

Although referring to a single section, these models are able to simulate, with various degrees of approximation, the behavior of the entire beam portion considered (Branson 1966; Beeby 1968; Rao and Subrahmanyam 1973; Gilbert and Warner 1978; Espion and Halleux 1988; "Building" 1989; Alwis 1990; Prakhya and Morley 1990; CEB 1993). Among such models, it is worth mentioning both the classical trilinear relationship [Fig. 3(a)] and some of the latest codes [i.e., Comité Euro-International du Béton (CEB) (1993)]. This allows $1/r$ to be computed as a suitable linear combination, through a γ coefficient, of curvatures $1/r_1$ and $1/r_2$, relating to Stage I and Stage II, respectively [Fig. 3(b)]

$$\frac{1}{r} = \gamma \cdot \frac{1}{r_1} + (1 - \gamma) \cdot \frac{1}{r_2} \quad (1)$$

Recently, various researchers moved their attention from wide portions of the beam to the block delimited by two consecutive flexural cracks. By means of modern experimental techniques that observe the whole surface of the beam (i.e., the geometric moiré technique used by Giuriani and Ronca 1979) researchers developed mechanical models of the entire block. These models (known as "block models") introduce a suitable bond-slip relationship and adopt simplified strain profiles in the cross sections. By assuming both the stress-strain relationships of steel and concrete to be known, the models impose the requirements of strain compatibility and equilibrium of forces in all sections of the block obtaining a system of nonlinear differential equations. Through suitable boundary conditions, the solution of this system makes it possible to

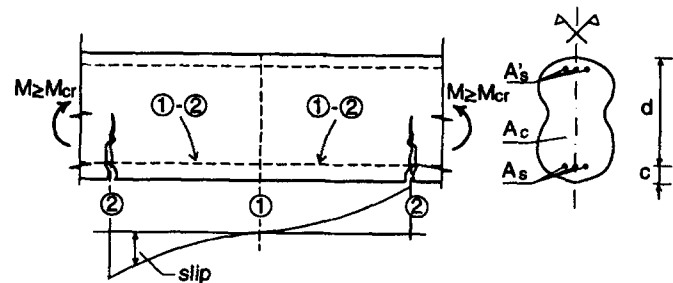


FIG. 1. Flexural Cracking in RC Beams

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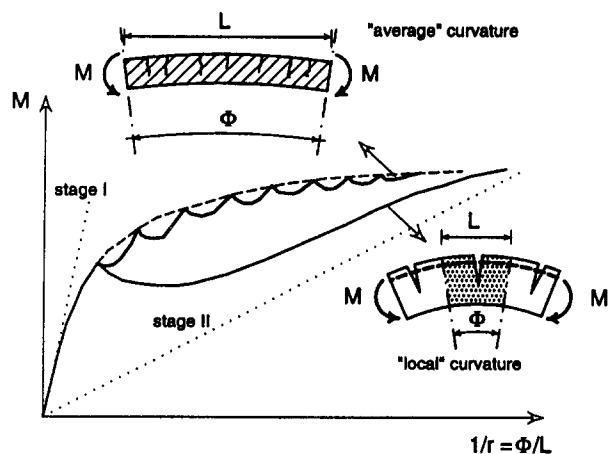


FIG. 2. Definition of Local and Average Moment-Curvature Relationship (Gelfi and Giuriani 1982)

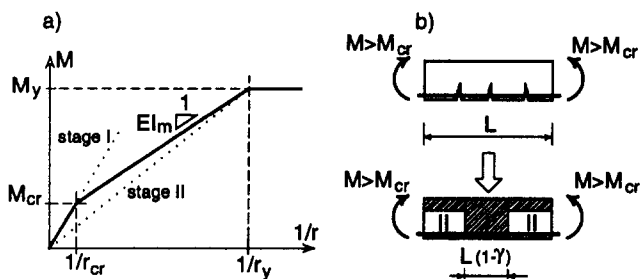


FIG. 3. Moment-Curvature Relationship: (a) Trilinear; (b) Combination of Curvatures of Stages I and II

determine, for the acting moment M , the states of stress, strain, slip, and curvature in each cross section (Westergaard 1933; Maldague 1965; Priestley et al. 1971; Gelfi and Giuriani 1982; Cohn and Riva 1987; Cosenza et al. 1991; Elieghausen and Li 1992; Sato et al. 1992; Creazza and Di Marco 1993; Vasiliev and Belov 1993).

In most cases, the block models assume a suitable length of the block rather than obtaining it as a solution of the problem. Moreover, the steel-concrete bond is evaluated by neglecting a possible unloading of bond stresses. The unloading phenomenon may occur, even with monotonically increasing moment, when the boundary conditions change due to the formation of a new crack (Ferretti et al. 1995). The aforementioned problems suggest the need for a block model that is able to take into account these effects in a more generalized way.

DEFINITION OF "GENERAL MODEL"

The model proposed in this paper occupies an intermediate position between two-dimensional finite-element models and the cross-sectional ones.

By means of appropriate boundary conditions, the entire beam is modeled through a succession of blocks divided by flexural cracks. In the cracked sections, the blocks interact via the concrete and the reinforcement. Hence, the problem appears to be statically indeterminate, and the solution would require a three-dimensional modeling of the entire beam (Vasiliev and Belov 1993). Nevertheless, the tests performed on statically determinate elements showed that the formation of a new crack does not modify significantly the stage of stress and strain in the other preexisting blocks (Giuriani and Ronca 1979). Accordingly, in the proposed model, the blocks are assumed to be mutually independent and delimited by cracks that start forming when the cracking moment M_{cr} is attained. In the cracked sections, the crack tip is assumed to be located on the neutral axis, and hence, Stage II assumptions are ap-

plied (Maldague 1965; Priestley et al. 1971; Giuriani and Plizzari 1989). The strain profile between the cracked section [Fig. 4(c)] and the uncracked one [Fig. 4(a)] is no longer plane but follows the bilinear pattern depicted in Fig. 4(b).

Although aiming to a simplified representation of physical behavior, the assumptions find confirmation both in laboratory tests (Giuriani and Ronca 1979; Jones et al. 1980) and in numerical analyses (Ngo and Scordelis 1967; Plauk and Hees 1981).

For a generic section of the block, together with the compatibility assumptions shown in Fig. 4, the equilibrium equations are considered, i.e.

$$\int_{A_c} \sigma_c dA_c + \sigma_s \cdot A_s + \sigma'_s \cdot A'_s = 0 \quad (2a)$$

$$\int_{A_c} \sigma_c \cdot y_c dA_c + \sigma_s \cdot y_s \cdot A_s + \sigma'_s \cdot y'_s \cdot A'_s = M(z) \quad (2b)$$

For a portion of the beam of infinitesimal length dz (Fig. 5) the equilibrium equation for the reinforcing bar alone can be written as

$$\frac{d\sigma_s}{dz} = \frac{p_s}{A_s} \cdot \tau[s(z)] \quad (3)$$

where A_s and p_s = area and perimeter, respectively, of the reinforcement in tension, while $\tau[s(z)]$ = bond stress.

Referring to the same portion of the beam, it is also possible to define the slip $s(z)$, corresponding to the difference in the displacements between two initially overlapping points belonging to steel and concrete

$$s(z) = s_s(z) = s'_{ct}(z) \quad (4)$$

Hence, by derivation with respect to z

$$\frac{ds}{dz} = \epsilon_s(z) - \epsilon'_{ct}(z) \quad (5)$$

where ϵ_s = strain of the reinforcing bar; and ϵ'_{ct} = tensile strain in the surrounding concrete (Fig. 5).

To solve the problem, the equilibrium and compatibility equations must be associated with the constitutive laws of the materials and with the bond-slip relationship τ - s .

In particular, since the study concerns the service load

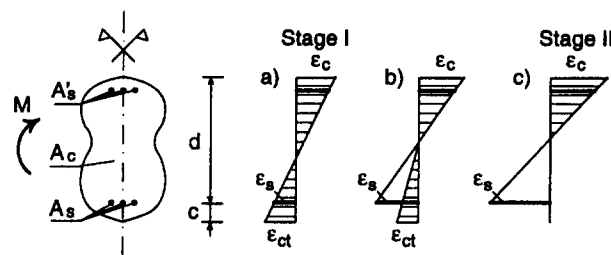


FIG. 4. Different Strain Profiles Assumed for Cross Sections: (a) Stage I; (b) Generic Cross Sections; (c) Stage II

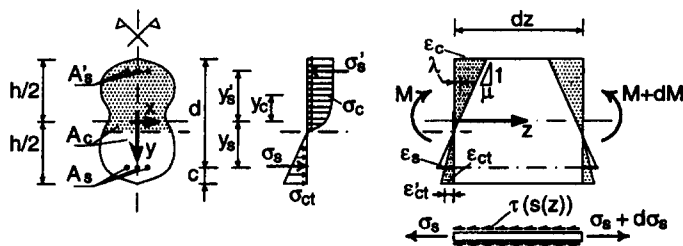


FIG. 5. Free-Body Diagrams

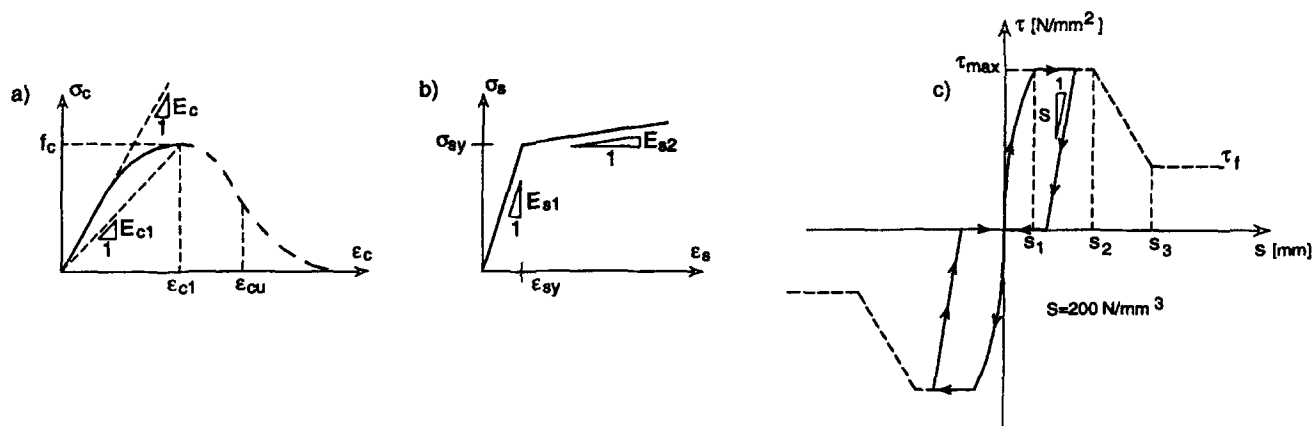


FIG. 6. Constitutive Laws Adopted (CEB 1993): (a) Stress-Strain Relationship for Concrete in Compression; (b) Stress-Strain Relationship for Reinforcing Steel; (c) Bond-Slip Relationship with Unloading Branches

range, the ascending branch [Fig. 6(a)] of the relationship proposed by CEB (1993) is considered as a reference for concrete in compression

$$\sigma_c = \frac{E_c/E_{c1} \cdot \epsilon_c/\epsilon_{c1} - (\epsilon_c/\epsilon_{c1})^2}{1 + (E_c/E_{c1} - 2) \cdot \epsilon_c/\epsilon_{c1}} \cdot f_c \quad (6)$$

The concrete in tension is assumed to have a linear elastic stress-strain relationship with modulus E_c .

The stress-strain relationship for reinforcing steel in tension or compression [Fig. 6(b)] is the following:

$$\sigma_s = E_{s1} \cdot \epsilon_s \quad \text{if } \epsilon_s \leq \epsilon_{sy} \quad (7a)$$

$$\sigma_s = \sigma_{sy} + E_{s2} \cdot (\epsilon_s - \epsilon_{sy}) \quad \text{if } \epsilon_s > \epsilon_{sy} \quad (7b)$$

For the τ - s relationship, again in accordance with CEB (1993), the following law is assumed:

$$\tau = \tau_{\max} \cdot \left(\frac{s}{s_1}\right)^\alpha \quad \text{if } 0 \leq s < s_1 \quad (8a)$$

$$\tau = \tau_{\max} \quad \text{if } s_1 \leq s < s_2 \quad (8b)$$

$$\tau = \tau_{\max} - (\tau_{\max} - \tau_f) \cdot \frac{(s - s_2)}{(s_3 - s_2)} \quad \text{if } s_2 \leq s < s_3 \quad (8c)$$

$$\tau = \tau_f \quad \text{if } s \geq s_3 \quad (8d)$$

with the proposed CEB (1993) reduction of bond stress near cracks. The unloading effects, which are normally produced by the gradually increasing number of cracks, are taken into account through the simplified cyclic relationship illustrated in Fig. 6(c).

For the solution of the problem, in addition to the foregoing equations, suitable boundary conditions are also adopted, that is, Stage II conditions for the cracked sections and Stage I conditions for the sections with perfect bond. From a mathematical viewpoint, (2)–(8) (which generally cannot be uncoupled), together with the boundary conditions defined for the edges of the block, constitute a differential problem called “two-point boundary value problem.”

In principle, this problem can be solved with the aid of various numerical techniques, such as relaxation through finite differences, finite-element methods, and shooting techniques (Press et al. 1992). For the numerical solution, the “multiple shooting method” was chosen by introducing, in the integration domain, two supporting sections where zero slip occurs. The position of such sections is shifted by trial and error until all of the boundary conditions are satisfied in the integration (Ferretti 1996).

The solution to the problem yields the distribution along the block of all of the unknowns static and kinematical: the state

of stress in concrete σ_c , σ_{ct} and steel σ_s , the slip s , the bond stress τ , and the curvature of each section μ . As a result the average curvature $1/r$, or the local curvature $1/r_{loc}$, can be determined as

$$\frac{1}{r} \text{ or } \frac{1}{r_{loc}} = \frac{1}{L} \cdot \int_L \mu \, dz = \frac{1}{L} \cdot \int_L \frac{\epsilon_s - \epsilon_c}{d} \, dz \quad (9)$$

where L = length of the considered reference portion (Fig. 2).

BEAM SUBJECTED TO CONSTANT BENDING MOMENT: COMPARISON WITH EXPERIMENTAL DATA

The proposed model was applied to the analysis of a constant moment beam (Fig. 7) tested by Giuriani and Sforza (1981) by means of a geometric moiré method. For this beam both the average and local moment-curvature experimental diagrams are known. Starting from the incipient first cracking ($M = M_{cr} = 1,350 \text{ N} \cdot \text{m}$), the numerical analysis examined the beam portion comprised between experimental cracks ③ and ④ [Fig. 7(a)] and offered the distribution of all the static and kinematical unknowns previously defined. In particular, the tensile stress σ_{ct} in concrete at the lower edge of the beam reaches the strength f_{ct} over an extensive portion in which a crack may theoretically originate. On the other hand, from the tests it has been determined that, in the portion under study, the cracks do not start simultaneously but, due to the inevitable, albeit small, scatter in the concrete tensile strength f_{ct} they open in the chronological order shown in Fig. 7(a).

Moreover the cracks are not evenly spaced, and their dis-

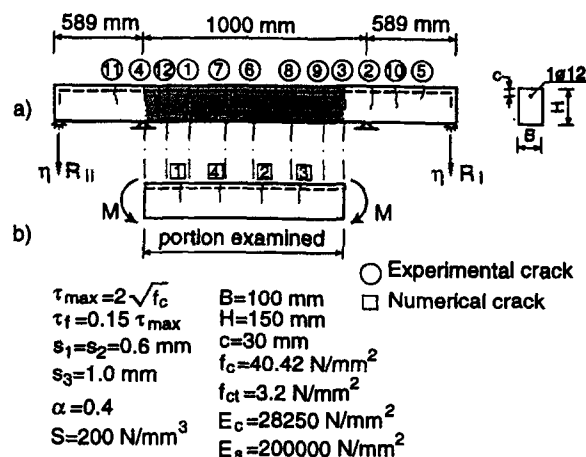


FIG. 7. Beam Tested by Giuriani and Sforza (1981): (a) Experimental Crack Pattern; (b) Analyzed Portion and Numerical Crack Pattern

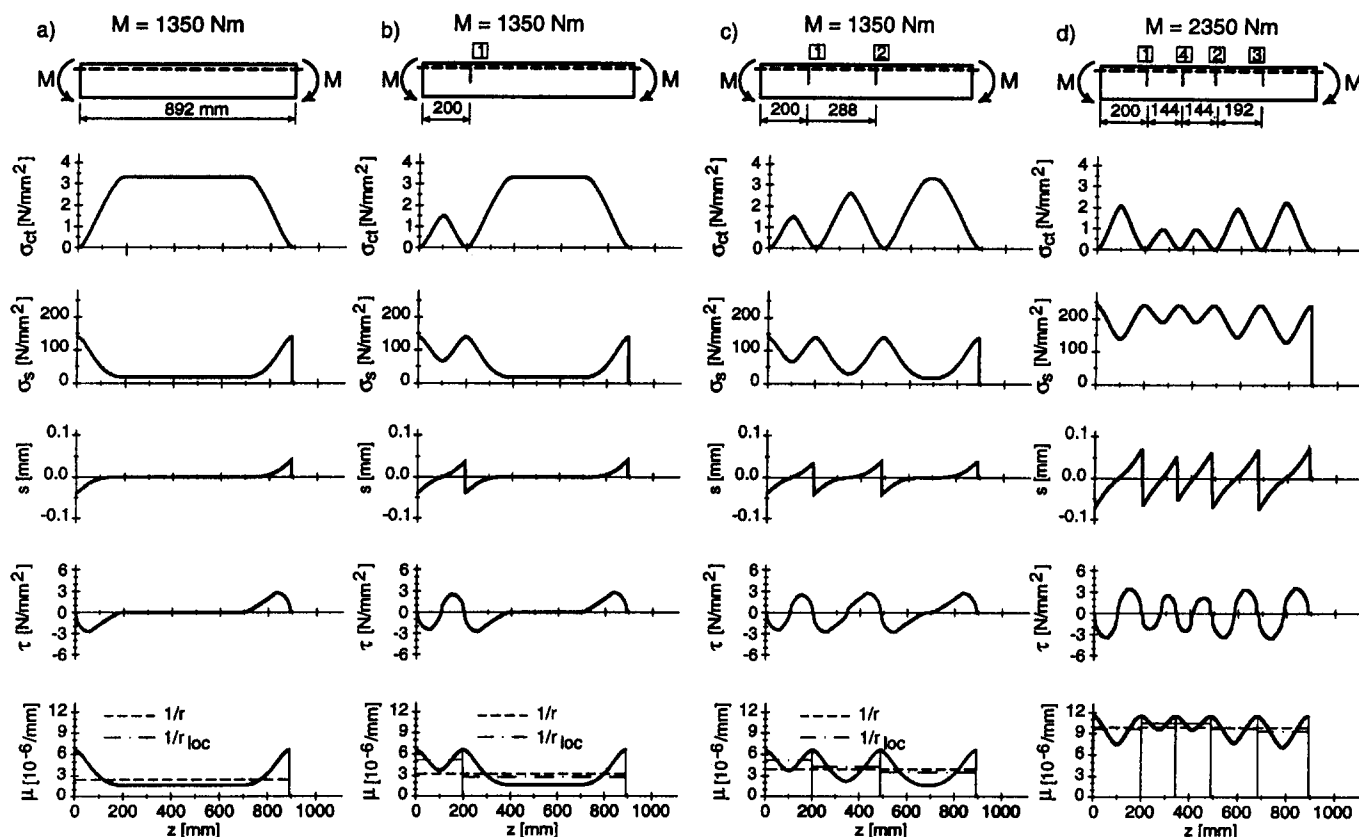


FIG. 8. Numerical Outcomes: (a) Incipient Cracking; (b) Formation of First Crack; (c) Formation of Second Crack; (d) Final Crack Pattern

tance varies randomly. For this reason the onset of the first [Fig. 8(b)] and second crack [Fig. 8(c)] is triggered in keeping with the test results. The two cracks reduce σ_{ct} and therefore the opening of a third crack can be attained only by increasing the bending moment. In this way the tensile stress σ_{ct} reaches f_{ct} only in one cross section, so that the position of the third crack [Fig. 8(c)] is univocally determined and virtually coincident with the experimental one [Fig. 7(a)]. The bending moment was increased until four cracks formed [Fig. 8(d)].

For the portion examined it now becomes possible to determine, from (9), the local $1/r_{loc}$ and the average $1/r$ curvatures by taking a suitable value of L (Fig. 9).

Local curvatures have been determined on the basis of the definition proposed by Giuriani and Sforza (1981), by considering L equal to the distance between the midway sections of

the neighboring cracked blocks. It can be observed that, for the same bending moment, local curvature values differ from one block to another, owing to the different distances between cracks. In Fig. 9, the values obtained for the average and local curvatures from the numerical procedure and the laboratory tests are compared. Their substantial agreement proves the validity of the proposed method. Furthermore, the small difference between local and average results confirms that, when dealing with members subjected to a virtually constant moment, the use of an average instead of a local relationship does not yield significant differences.

BEAM SUBJECTED TO VARIABLE BENDING MOMENT: COMPARISONS WITH EXPERIMENTAL DATA

The proposed model was also applied to the analysis of a beam with variable bending moment tested by Gelfi and Giuriani (1982) by means of the geometric moiré method. The main geometric and mechanical properties of the beam examined by Gelfi and Giuriani are shown in Fig. 10.

The numerical investigation, performed by increasing the load R_1 , showed that when $R_1 = 2,150$ N the tensile strength of the concrete is reached on the support section [Fig. 11(a)], where crack 1 starts, in perfect agreement with the test results [Fig. 11(b)].

The crack formation produces an instantaneous reduction of concrete tensile stress σ_{ct} , as can be evinced from a comparison between Fig. 11(a), relating to the incipient cracking condition, and Fig. 11(b) showing the first crack formation. The same figure shows that the new stress distribution does not reach f_{ct} , so that it becomes necessary to increase the load to reach incipient cracking conditions and the formation of crack 2 [Fig. 11(c)]. By increasing the load gradually, the five cracks illustrated in Fig. 10(b) and in Fig. 11(d) are obtained.

Thus it is possible to compare the numerical crack pattern

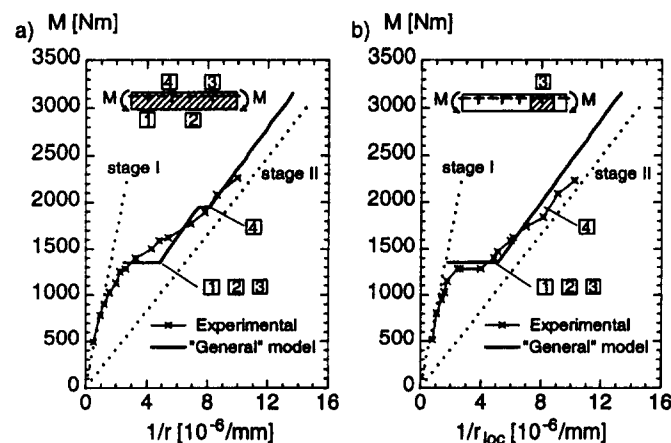


FIG. 9. Comparison between General Model and Experimental Data (Giuriani and Sforza 1981): (a) Average Moment-Curvature Curves; (b) Local Moment-Curvature Curves

[Fig. 10(b)] with the one detected in the laboratory tests [Fig. 10(a)]. The good agreement between the two results confirms the capability of the model to simulate the complex physical behavior also in terms of the crack history.

Compared with constant moment conditions the presence of shear produces a more regular crack pattern, with nearly even spacing of the cracks, in accordance with the test results (Gelfi and Giuriani 1982). The presence of shear therefore seems to neutralize the uncertainty into the position of cracks, which are now univocally positioned.

Referring to Fig. 11 it may be noted that the marked variation in μ points out the need for a local evaluation of the

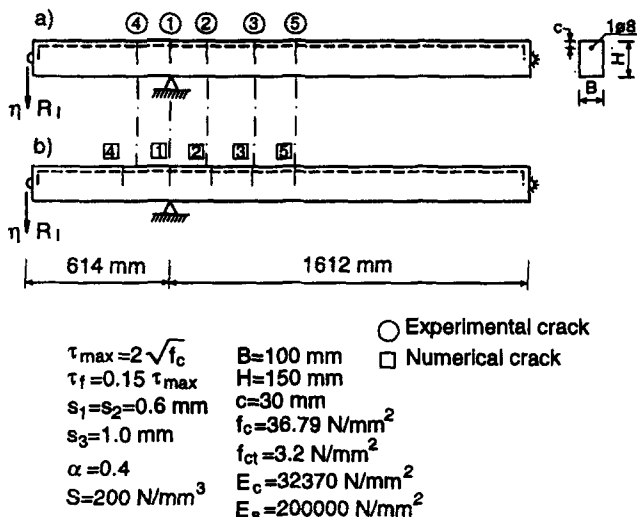


FIG. 10. Beam Tested by Gelfi and Giuriani (1982): (a) Experimental Crack Pattern; (b) Analyzed Portion and Relative Crack Pattern

curvature for each individual cracked block. As a matter of fact, an average evaluation does not appear to be accurate. As for the constant moment beam, the local curvature is determined on the basis of (9) assuming L equal to the distance between the midsections of two neighboring blocks, while M is evaluated in the cracked section. Thus, the definition is the same as the one used in the testing campaign (Gelfi and Giuriani 1982). Fig. 12 shows the local moment-curvature diagrams, for the three cracked blocks, originated by the tests and by the numerical analysis. The three blocks display virtually identical deformation characteristics and good agreement with the experimental results.

ENVELOPE OF POTENTIAL SOLUTIONS: DEFINITION OF "RANGE MODEL"

As for the crack configuration observed in the beam subjected to a constant moment, a deeper investigation is required, because the cracks may occur in positions other than the selected ones.

This represents a fundamental aspect of the problem, that is, the practical impossibility of univocal definition of the crack pattern evolution. In fact, as can be seen in Fig. 8(a), when the initial cracking stage is attained ($M = M_{cr}$), σ_{ct} reaches the ultimate value f_{ct} in a wide portion of the beam. This justifies the opening of the first crack in any section of this portion.

To take into account this phenomenon, the numerical analysis was repeated by assuming three different possible developments of the crack pattern.

The results are illustrated in Fig. 13(a) and clearly show that the moment-curvature relationships are affected by the variations of the initial crack pattern.

This type of behavior, as pointed out by Ferretti et al. (1995), is not a prerogative of beams subjected to constant

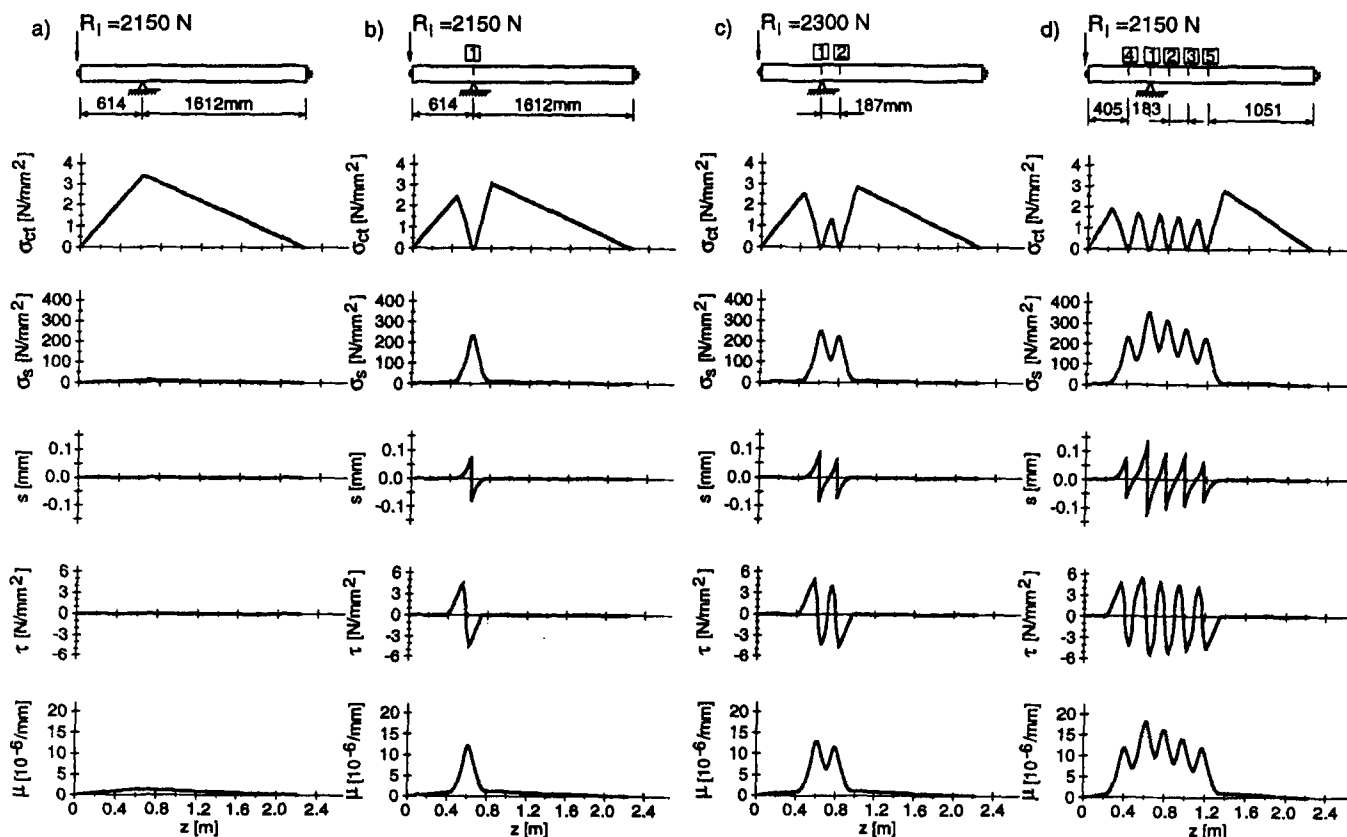


FIG. 11. Numerical Outcomes: (a) Incipient Cracking; (b) Formation of First Crack; (c) Formation of Second Crack; (d) Final Crack Pattern

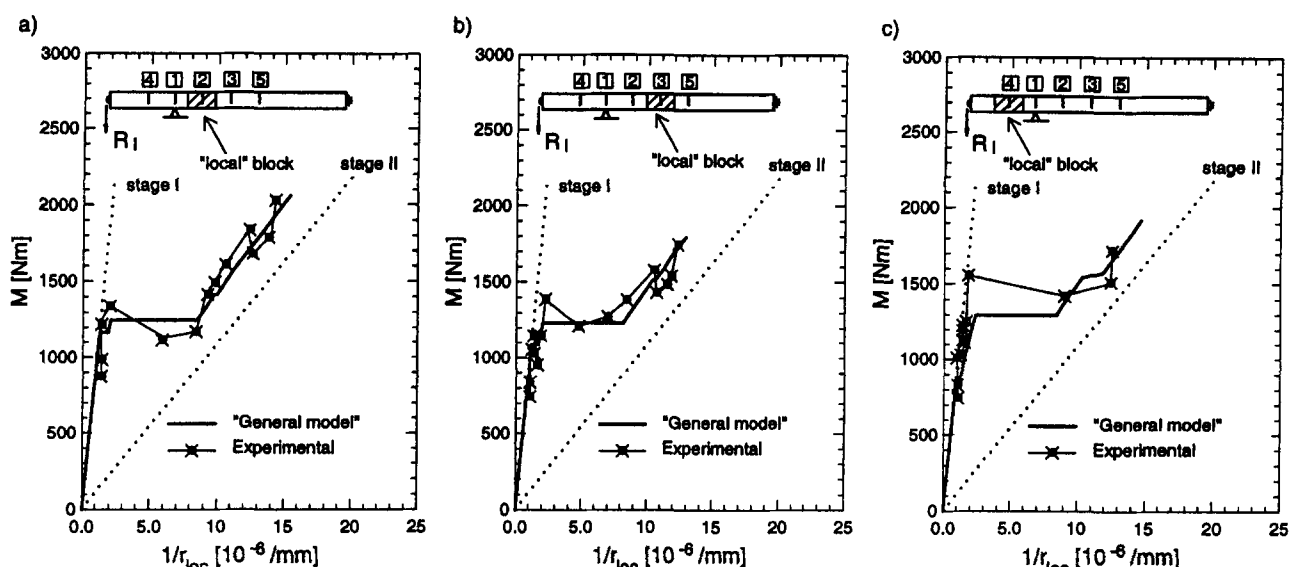


FIG. 12. Comparison between Experimental (Gelfi and Giuriani 1982) and Numerical $M-1/r_{loc}$ Diagrams: (a) Local Block 2 (b) Local Block 3; (c) Local Block 4

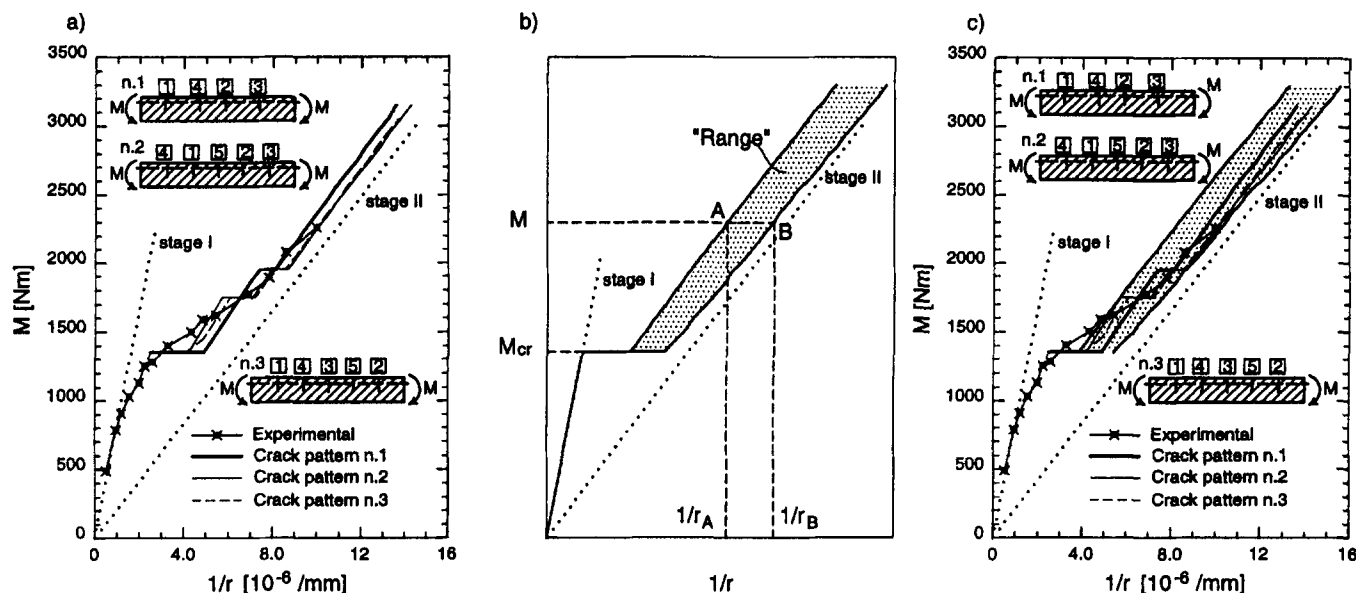


FIG. 13. Uncertainty of Crack Pattern: (a) Three Possible Outcomes; (b) Range Model; (c) Capability of Range Model to Include Numerical Curves and Experimental Data

bending moment but is also observed in RC tensile members (Avalle et al. 1994) and, in general, in all the situations where the initial crack pattern is uncertain. Such situations surely include also members subjected to variable bending moment. In fact, in such members the crack pattern is no longer affected by the same uncertainty associated with a constant moment beam. However, the presence of stirrups or the load history may alter the theoretical regular evolution of the cracks (Rizkalla and Hwang 1984). To deal with this random nature of the phenomenon, instead of pursuing a probabilistic approach, all of the solutions due to different possible crack evolution hypotheses could be included in a well-defined field or "range." This range, which will be defined in terms of a moment-curvature relationship, may be delimited by the curves of maximum and minimum deformability. The $M-1/r$ relationships obtained from all of the possible crack patterns must therefore fall within this range. The definition of the range, on the basis of all the infinite possible solutions, cannot be proposed. However the observation of the beams suggests that, for a given moment M , the deformability of the generic block falls within the deformability of the block in incipient cracking

conditions and that of the block produced by its cracking. Therefore, the deformability of the member is within these two limits. For these reasons, the identification of the so-called range can be achieved through the study of deformability of an individual block in Condition A (incipient cracking) and in Condition B [after the opening of the crack; Fig. 13(b)]. With reference to Fig. 14(a), Condition A is characterized by a block length $L = 2l_r$ (l_r being the so-called transfer length) so that $\sigma_{ct} = f_{ct}$ in the midspan of the block. Conversely, in Condition B, the previous block is assumed to crack just in the midspan, and the deformability is computed for a half-block length $L = l_r$ subjected to the same M . Therefore the identification of the range assumes the the block length $L = 2l_r$ varies continuously as a function of the bending moment M .

With reference to the generic point A, it is possible to consider a block with length $2l_r$ [Fig. 14(a)] comprised between two adjacent cracks and subjected to $M \geq M_{cr}$. The behavior of such a block can be described through the same equilibrium and compatibility conditions used for the "general model" (2)–(8), but with different boundary conditions. In particular, for the cracked sections [denoted with ② in Fig. 14(a)] Stage

II conditions are assumed to apply. For the midspan section the incipient cracking conditions requires $\sigma_{cr} = f_{cr}$, while $s = 0$ due to symmetry. From a mathematical viewpoint, this problem represents a particular boundary value problem, whose solution requires the determination of the unknown length l_{tr} . This length is computed with a trial-and-error algorithm start-

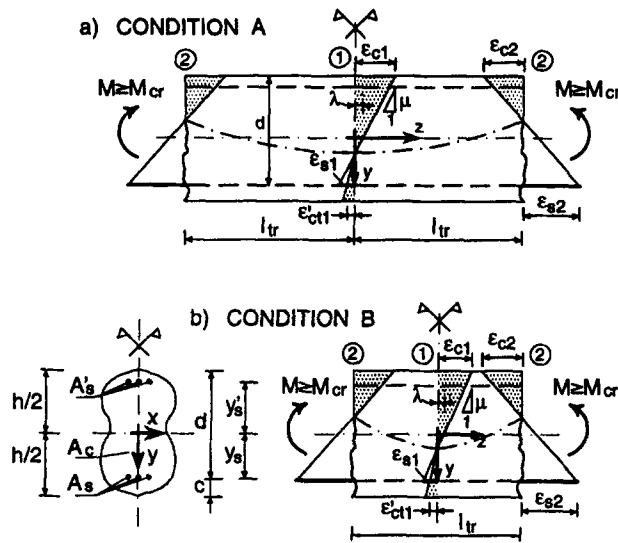


FIG. 14. Block Considered to Define Range: (a) Incipient Cracking; (b) Formation of Crack

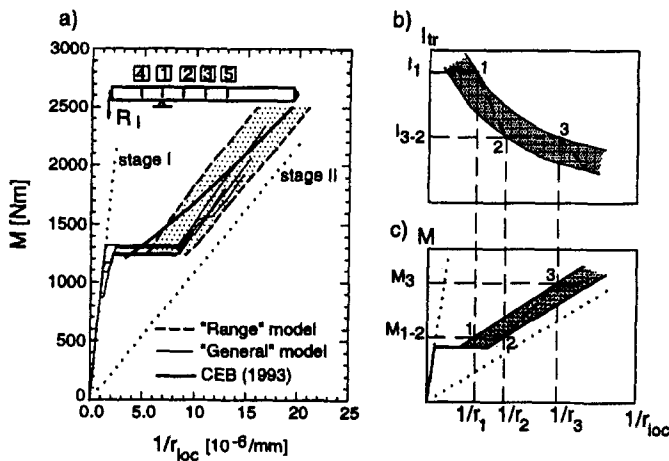


FIG. 15. $M-1/r$ Relationships Relating to Beam Examined by Gelfi and Giuriani (1982): (a) Comparison between Range Model, General Model, and CEB (1993); (b, c) Continuous and Discrete Development of Transfer Length l_{tr}

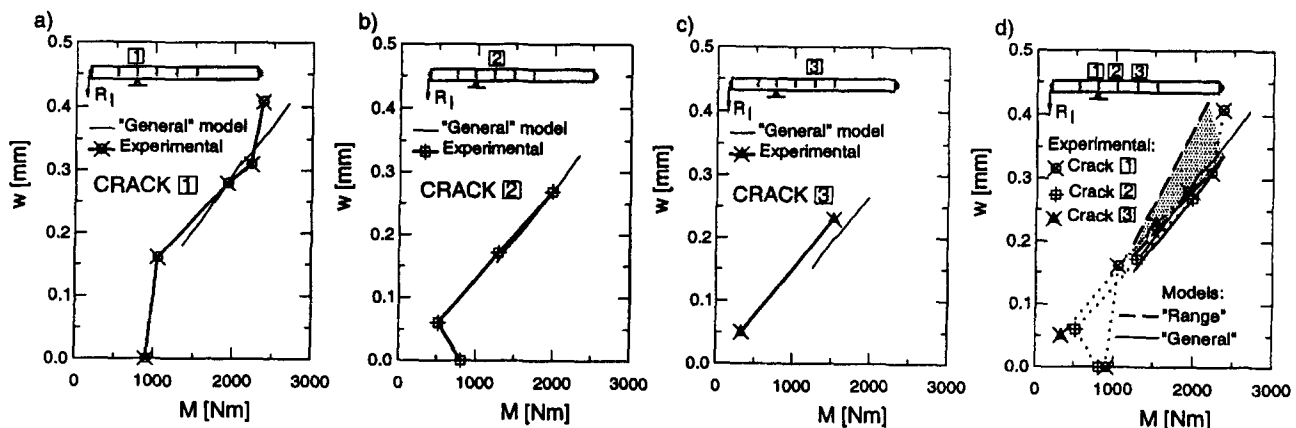


FIG. 16. Evaluation of Crack Width w as Function of Moment M : (a-c) Comparison between General Model Curves and Test Data (Gelfi and Giuriani 1982); (d) Comparison between General Model Curves, Range Model Curves and Test Data (Gelfi and Giuriani 1982)

ing with the integration of the differential equations from the midsection and shifting the cracked section until all the boundary conditions are satisfied.

On the other hand, point B [Fig. 13(b)] is determined assuming that, when the previous incipient crack occurs, two separate blocks of known length l_{tr} are formed. Under these circumstances, the boundary conditions in the cracked sections (denoted with ②) are still those of Stage II. Instead, in the midspan section of the new block, $s = 0$ and the stress σ_{cr} is still unknown. The boundary conditions, relating to half the block, define a two-point boundary value problem, which can be solved through the same shooting method described for the general model. The range identified in this way is sufficient to encompass the numerous possible solutions [Fig. 13(c)]. For the previous constant moment example, the $M-1/r$ diagrams produced by the test and the three general analyses [Fig. 13(a)] are depicted in Fig. 13(c).

The range, obtained through the local analysis of the same beam, is able to contain the $M-1/r_{loc}$ diagrams, both in the constant moment case illustrated previously and in the case of variable moment. Fig. 15(a), for instance, shows that the local diagrams of the variable moment beam (previously analyzed by the general model) fall within the range. Fig. 15(a) also shows the $M-1/r$ curve according to the CEB (1993) model. Except for the initial portion, this curve, obtained by means of average observations, falls within the range, although the latter has been obtained according to a local approach. Thus, it can be observed that, in this particular case and for values of M slightly higher than M_{cr} , the indications given in CEB (1993) are not reliable when dealing with all those structural conditions that require a local evaluation of the curvature. As a final consideration, it should be pointed out that, in the common block model approaches, the crack pattern evolves stepwise with increasing applied moment, that is, by halving gradually the length of each cracked portion [e.g., Creazza and Di Marco (1993)]. Conversely, by means of the range model, a continuous variation of the block length L as a function of the curvature is obtained [Fig. 15(b)]. However the stepwise evolution, which reflects the physical behavior, is still contained in the range. Fig. 15(b) illustrates the stepwise development of the transfer length l_{tr} and the corresponding shift of the bending moment from $M_{1,2}$ to M_3 . Diagrams substantially similar to the one shown in Fig. 15(b) have also been proposed for tensile members by Bažant and Oh (1982), Floegl and Mang (1982), Tanabe and Yoshikawa (1987), and Wu et al. (1991).

In the determination of both the deformability range and the general model, the solution of the differential problem gives the distribution of the slip s , whose values identify the width w of the cracks. In this way, it was possible to plot the $w-M$

curves, with reference to the structural member shown in Fig. 10. In particular, in Figs. 16(a–c) the experimental data relating to three different cracks and the numerical results obtained with the general model are compared.

Together with the “deformability range,” it is also possible to obtain the “crack width range” shown in Fig. 16(d), which nearly encompasses the test data and the numerical results of the general model. A similar range for the crack widths was proposed, in the case of RC tensile members, by Somayaji and Shah (1981). The “range approach” seems to be able to quantify the “model uncertainties” arising from the randomness of equipotential crack patterns, characterized by the same concrete tensile strength instead of its statistical variation along the member.

CONCLUSIONS

Some important features of the deformability of RC beams subjected to constant or variable bending moment can be explained by the proposed general model. A few considerations can be finally remarked.

Fig. 15(a) confirms that the moment-curvature models proposed in CEB (1993) are average-type models. The reason for this should probably be sought in the direct dependence of the CEB guidelines on the early experimental observations conducted by testing methods focusing on average characteristics. In some situations, the average approach is clear-cut, and, as already pointed out by Giuriani and Ronca (1979), it may lead to rough structural analyses.

- In these situations the local approach seems to work better (Fig. 12). However both approaches are affected by the indetermination of the initial crack pattern.
- For this reason a range approach, able to encompass all the possible deformabilities has been proposed [Fig. 13(c)].

It should be observed that the introduction of the range could not be considered as a direct tool for design practice, which is usually carried out by means of simplified models. The identification of a range of possible real situations, however, enables the designer to check the validity of the adopted simplified M - $1/r$ relationship and to quantify the uncertainty of the parameters [e.g., coefficient γ in (1)].

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

A = area;
 c = cover;
 d = effective height of beam;
 E = modulus of elasticity;
 f = strength of material;
 L = length of element or block;
 l_r = transfer length;
 M = bending moment;
 M_{cr} = cracking moment;
 n = number of integration points;
 p = perimeter of reinforcing bar;
 s = slip or displacement;

w = crack width;
 y = vertical coordinate;
 z = horizontal coordinate;
 $1/r$ = curvature;
 $\varnothing$ = diameter of reinforcing bar;
 ϵ = strain;
 μ = cross-sectional curvature;
 σ = axial stress;
 τ = bond stress; and
 Φ = rotation.

Subscripts

c = concrete in compression;
 ct = concrete in tension;
 $'ct$ = concrete in tension surrounding reinforcing bar;
 I = stage I;
 II = stage II;
 loc = local observation;
 s = steel in tension;
 $'s$ = steel in compression; and
 y = yielding.